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Ruin probability for Gaussian integrated processes

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Abstract

Pickands constants play an important role in the exact asymptotic of extreme values for Gaussian stochastic processes. By the *generalized Pickands constant* $\mathscr{H}_\eta$ we mean the limit

$$\mathscr{H}_\eta = \lim_{T\to\infty} \frac{\mathscr{H}_\eta(T)}{T},$$

where $\mathscr{H}_\eta(T) = \mathbb{E}\exp(\max_{t\in[0,T]}(\sqrt{2}\eta(t) - \sigma_\eta^2(t)))$ and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$.

Under some mild conditions on $\sigma_\eta^2(t)$ we prove that $\mathscr{H}_\eta$ is well defined and we give a comparison criterion for the generalized Pickands constants. Moreover we prove a theorem that extends result of Pickands for certain stationary Gaussian processes.

As an application we obtain the exact asymptotic behavior of $\psi(u) = \mathbb{P}(\sup_{t\geqslant 0}\zeta(t) - ct > u)$ as $u \to \infty$, where $\zeta(x) = \int_0^x Z(s)\,\mathrm{d}s$ and $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t)$ fulfilling some integrability conditions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Pickands III (1969a, b) found an elegant way of computing the exact asymptotics of the probability $\mathbb{P}(\max_{t\in[0,T]} X(t) > u)$ for a centered stationary Gaussian process $X(t)$

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with covariance function $R(t)=1-|t|^{\alpha}+o(|t|^{\alpha})$ as $t\to 0$, $\alpha\in(0,2]$ and $R(t)<1$ for all $t>0$. For such a process he proved

$$\mathbb{P}\left(\max_{t\in[0,T]} X(t)>u\right)=\mathscr{H}_{B_{\alpha/2}}Tu^{2/\alpha}\Psi(u)(1+o(1)) \quad \text{as } u\to\infty, \tag{1.1}$$

where $\mathscr{H}_{B_{\alpha/2}}$ is the *Pickands constant* and $\Psi(u)$ is the tail distribution of standard normal law. Recall that $\mathscr{H}_{B_{\alpha/2}}$ is defined by the following limit:

$$\mathscr{H}_{B_{\alpha/2}}=\lim_{T\to\infty}\frac{\mathbb{E}\exp(\max_{t\in[0,T]}\sqrt{2}B_{\alpha/2}(t)-\mathrm{Var}(B_{\alpha/2}(t)))}{T}, \tag{1.2}$$

where $B_{\alpha/2}(t)$ is a fractional Brownian motion (FBM) with Hurst parameter $\alpha/2$, that is a centered Gaussian process with stationary increments, continuous sample paths and variance function $\mathrm{Var}(B_{\alpha/2}(t))=t^{\alpha}$. Pickands proved (1.1) using *the double sum method*, that is by breaking the interval $[0,T]$ into several subintervals on which the following asymptotics may be applied: for each $T>0$

$$\mathbb{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t)>u\right)=\mathscr{H}_{B_{\alpha/2}}(T)\Psi(u)(1+o(1)) \tag{1.3}$$

as $u\to\infty$, where

$$\mathscr{H}_{B_{\alpha/2}}(T)=\mathbb{E}\exp\left(\max_{t\in[0,T]}(\sqrt{2}B_{\alpha/2}(t)-\mathrm{Var}(B_{\alpha/2}(t)))\right). \tag{1.4}$$

Asymptotics (1.3) is a useful tool for computing the exact asymptotics in extreme value theory for a wide class of Gaussian processes (see Piterbarg, 1996). Unfortunately it does not cover all the cases interesting in applications (see for example the class of Gaussian integrated processes considered in Dębicki, 1999). In particular the stationarity assumption seem to be too strong. We present an extension of (1.3) in Section 2 (Theorem 2.1).

It turns out that the asymptotics obtained in Theorem 2.1 yields a natural extension of Pickands constants. Namely instead of FBM $B_{\alpha/2}(t)$ in (1.2) there appear more general centered Gaussian processes $\eta(t)$ with stationary increments.

Throughout this article $\eta(t)$ is a centered Gaussian process with stationary increments, a.s. continuous sample paths, $\eta(0)=0$ and such that the variance function $\mathrm{Var}(\eta(t))=\sigma_\eta^2(t)$ satisfies

C1 $\sigma_\eta^2(t)\in C^1([0,\infty))$ is strictly increasing and there exists $\varepsilon>0$ such that

$$\limsup_{t\to\infty}\frac{t\dot\sigma_\eta^2(t)}{\sigma_\eta^2(t)}\leqslant\varepsilon; \tag{1.5}$$

C2 $\sigma_\eta^2(t)$ is regularly varying at 0 with index $\alpha_0\in(0,2]$ and $\sigma_\eta^2(t)$ is regularly varying at ∞ with index $\alpha_\infty\in(0,2)$.

In the paper we use the notation $\dot\sigma^2(t)$ or $\ddot\sigma^2(t)$ for the derivatives of $\sigma^2(t)$.

Note that C1 is strongly related to C2. In fact if $\sigma_\eta^2(t)$ satisfies C1 in such a way that $\lim_{t\to\infty}\sigma_\eta^2(t)=\infty$ and $\lim_{t\to\infty}t\dot\sigma_\eta^2(t)/\sigma_\eta^2(t)=\varepsilon$, then $\sigma_\eta^2(t)$ is regularly varying at ∞ and $\alpha_\infty = \varepsilon$ (see Bingham et al., 1987, p. 59). Conversely if $\sigma_\eta^2(t)$ is regularly varying at ∞ and $\dot{\sigma}_\eta^2(t)$ is ultimately monotone, then (1.5) holds.

For $\eta(t)$ that satisfies C1–C2 define

$$\mathscr{H}_\eta(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]}(\sqrt{2}\eta(t) - \sigma_\eta^2(t))\right). \tag{1.6}$$

More generally for independent centered Gaussian processes with stationary increments $\eta_1(t),\dots,\eta_N(t)$ that satisfy C1–C2, where the indices of regularity of variance functions may differ for each process, we define

$$\mathscr{H}_{\eta_1,\dots,\eta_N}(T) = \mathbb{E}\exp\left(\max_{(t_1,\dots,t_N)\in[0,T]^N}\left(\sqrt{\frac{2}{N}}\sum_{i=1}^N \eta_i(t_i) - \frac{1}{N}\sum_{i=1}^N \sigma_{\eta_i}^2(t_i)\right)\right). \tag{1.7}$$

Note that in a special case, when $\eta(t) = B_{\alpha/2}(t)$ and $N = 1$, we obtain the constants $\mathscr{H}_{B_{\alpha/2}}(T)$ defined in (1.4). We analyze properties of $\mathscr{H}_{\eta_1,\dots,\eta_N}(T)$ in Section 3.

By *the generalized Pickands constant* $\mathscr{H}_\eta$ we understand

$$\lim_{T\to\infty}\frac{\mathscr{H}_\eta(T)}{T} = \mathscr{H}_\eta,$$

provided that the limit exists. In Section 3 (see Theorem 3.1) we prove that under conditions C1–C2 this limit exists, is positive and finite. Moreover in Theorem 3.2 we give a comparison criterion for generalized Pickands constants.

With $\eta(t)$ we associate a family $\{X_{\eta;u}(t), u > 0\}$ (indexed by $u > 0$) of centered Gaussian processes, where the relation between $\eta(t)$ and $X_{\eta;u}(t)$ is given by assumption D0 presented in Section 2. By the attached bar we always mean the standardized process, that is $\bar{X}(t) = X(t)/\sigma_X(t)$.

In Section 2 (Theorem 2.1) we extend asymptotics (1.3) to a standardized family of Gaussian fields $\{\bar{X}_{\eta;u}(t), u > 0\}$ that satisfy condition D0.

Combination of Theorem 2.1 with the double sum method yields new exact asymptotics in extreme value theory. In particular in Section 4 we present Theorem 4.1 which extends results of Piterbarg, 1996 and enables us to obtain exact asymptotics for some families of Gaussian processes $\{X_{\eta;u}(t), u > 0\}$, where for sufficiently large u the variance function $\sigma_{X_{\eta;u}}^2(t)$ attains maximum at a unique point t_u.

Recently the asymptotics of

$$\psi(u) = \mathbb{P}\left(\sup_{t\geqslant 0}\zeta(t) - ct > u\right)$$

for a centered Gaussian process $\zeta(t)$ with stationary increments and $c > 0$ was studied in many papers; see e.g. Norros (1994), Dębicki and Rolski (1995, 2000), and Kulkarni and Rolski (1994). The problem of analyzing $\psi(u)$ stemmed from the theory of Gaussian fluid models, where the following cases are of special interest:

- $\zeta(x) = \int_0^x Z(s)\,\mathrm{d}s$, where $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t) = \mathbb{E}Z(0)Z(t)$ fulfilling some integrability conditions; we call such the case *integrated Gaussian* (IG),
- $\zeta(x) = B_{\alpha/2}(t)$ being a fractional Brownian motion with Hurst parameter $\alpha/2$, where $\alpha \in (0,2)$.

The last model was recently studied by Hüsler and Piterbarg (1999) who obtained exact asymptotic of $\psi(u)$ for $\zeta(x)$ being a fractional Brownian motion; see also Narayan (1998). Theorem 4.1, presented in Section 4, enables us to obtain the exact asymptotics of $\psi(u)$ for a class of IG processes that play an important role in the fluid model theory and is not covered by the results of Hüsler and Piterbarg (1999). Namely we focus on the case where $\zeta(x)=\int_0^x Z(s)\,\mathrm{d}s$ possesses the *short range dependence* (SRD) property, that is the covariance function $R(t)$ of $Z(t)$ fulfills

SRD.1 $R(t)\in C([0,\infty))$, $\lim_{t\to\infty} tR(t)=0$;

SRD.2 $\int_0^t R(s)\,\mathrm{d}s>0$ for each $t>0$ and $t=\infty$;

SRD.3 $\int_0^\infty s^2|R(s)|\,\mathrm{d}s<\infty$.

We exclude from the following considerations the degenerated case $R(0)=0$. We comment on assumptions SRD and give the exact asymptotic of $\psi(u)$ for $\zeta(t)\in$ SRD in Section 5.

2. Extension of Pickands theorem

We write $\{X_{\eta;u}(t),u>0\}$ for the family of centered Gaussian processes $\{X_{\eta;u}(t): t\geqslant 0\}$ $(u>0)$ and assume that for each $u>0$ the Gaussian process $X_{\eta;u}(t)$ has continuous trajectories. The family $\{X_{\eta;u}(t),u>0\}$ is related to a Gaussian process $\eta(t)$ with stationary increments and variance function $\sigma_\eta^2(t)$ that satisfies C1–C2 in such a way that the following assumption holds

D0 There exist functions $\Delta(u)$ and $f(u)$ such that

$$\sup_{s,t\in J(u)}\left|\frac{1-\mathrm{Cov}(\bar{X}_{\eta;u}(t),\bar{X}_{\eta;u}(s))}{\sigma_\eta^2(|t-s|)/f^2(u)}-1\right|\to 0$$

as $u\to\infty$, where $J(u)=[-\Delta(u),\Delta(u)]$ and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$ that satisfies C1–C2.

Remark 2.1. The assumption that $\sigma_\eta^2(t)$ is strictly increasing ensures that asymptotically (for large u) $\mathrm{Cov}(\bar{X}_{\eta;u}(t),\bar{X}_{\eta;u}(s))$ is a decreasing function of $|t-s|$ for $s,t\in J(u)$. It plays a crucial role in the technique of the proof of Theorems 3.1 and 4.1 (Lemmas 6.1 and 6.2).

In the sequel we present families of Gaussian processes that satisfy D0.

Example 2.1. *Assumption* D0 *covers the class of Gaussian processes analyzed by Pickands* III (1969a). *Namely let $X(t)$ be a stationary centered Gaussian process with covariance function $R(t)$ such that $R(t)=1-|t|^\alpha+o(|t|^\alpha)$ as $t\to 0$ ($\alpha\in(0,2]$). Straightforward calculation shows that $X(t)$ satisfies* D0 *with $\eta(t)=B_{\alpha/2}(t)$ (and thus $\sigma_\eta^2(t)=|t|^\alpha$), $\Delta(u)$ such that $\lim_{u\to\infty}\Delta(u)=0$ and $f(u)=1$. This immediately implies that, for a given function $h(u)>0$, the family $\{X_{\eta;u}(t)=X(t/h(u)),u\}$ satisfies* D0 *with $\eta(t)=B_{\alpha/2}(t)$, $\Delta(u)$ such that $\lim_{u\to\infty}\Delta(u)/h(u)=0$ and $f(u)=h^{\alpha/2}(u)$.*

Presented in this section Theorem 2.1 enables us to relax assumption that $X(t)$ in (1.3) is stationary (which in view of further considerations is too restrictive) and to give the exact asymptotic for families of Gaussian processes $\{\bar{X}_{\eta;u}(t)\}$ for which the structure of covariance function changes with parameter u. In Example 2.2 we present such a family. This family plays a crucial role in the rest of the paper.

Example 2.2. *Consider a centered Gaussian process* $\zeta(t)$ *with stationary increments and the variance function* $\sigma_\zeta^2(t)$ *that satisfies* C1–C2. *Define* $X_{\eta;u}(t)=\zeta(h(u)+t)$ *where* $h(u)$ *is such that* $\lim_{u\to\infty} h(u)=\infty$. *In the following lemma we show how the structure of covariance function of* $\bar{X}_{\eta;u}(t)$ *depends on* u, *appropriately satisfying* D0.

Lemma 2.1. *If* $\zeta(t)$ *is a centered Gaussian process with stationary increments that satisfies* C1–C2, *then for* $h(u)$ *such that* $\lim_{u\to\infty} h(u) = \infty$, *the process* $X_{\eta;u}(t) = \zeta(h(u) + t)$ *satisfies* D0 *with* $f(u) = \sqrt{2}\sigma_\zeta(h(u))$, $\eta(t) = \zeta(t)$ *and* $\Delta(u)$ *such that* $\lim_{u\to\infty} \Delta(u)/h(u) = 0$.

Proof. Let $s < t$. From the definition of $X_{\eta;u}(t)$

$$\mathrm{Cov}(\bar{X}_{\eta;u}(t), \bar{X}_{\eta;u}(s)) - 1 = \frac{(\sigma_\zeta(h(u)+t) - \sigma_\zeta(h(u)+s))^2}{2\sigma_\zeta(h(u)+s)\sigma_\zeta(h(u)+t)} - \frac{\sigma_\zeta^2(|t-s|)}{2\sigma_\zeta(h(u)+s)\sigma_\zeta(h(u)+t)} = W_1 - W_2. \tag{2.1}$$

Since $\sigma_\zeta(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0,2)$ it suffices to show that $W_1/W_2 \to 0$ uniformly for $s,t \in [-\Delta(u), \Delta(u)]$ as $u \to \infty$. It follows from the fact that for sufficiently large u

$$\frac{W_1}{W_2} = \frac{(\sigma_\zeta(h(u)+t) - \sigma_\zeta(h(u)+s))^2}{\sigma_\zeta^2(|t-s|)} = \frac{(\sigma_\zeta^2(h(u)+t) - \sigma_\zeta^2(h(u)+s))^2}{\sigma_\zeta^2(|t-s|)(\sigma_\zeta(h(u)+t)+\sigma_\zeta(h(u)+s))^2}$$

$$\leqslant \frac{(\sigma_\zeta^2(h(u)+t) - \sigma_\zeta^2(h(u)+s))^2}{4\sigma_\zeta^2(|t-s|)\sigma_\zeta^2(h(u)-\Delta(u))}$$

$$= \frac{1}{4}\left(\frac{|t-s|\dot{\sigma}_\zeta^2(h(u)+\rho)}{\sigma_\zeta(|t-s|)\sigma_\zeta(h(u)-\Delta(u))}\right)^2 \tag{2.2}$$

$$\leqslant \varepsilon^2 \left(\frac{\sigma_\zeta(h(u)+\rho)}{(h(u)+\rho)} \frac{|t-s|}{\sigma_\zeta(|t-s|)}\right)^2, \tag{2.3}$$

where from the mean value theorem there exists $\rho \in [s,t]$ such that (2.2) is satisfied. Inequality in line (2.3) is a consequence of the fact that by C1, for sufficiently large u, there exists $\varepsilon > 0$ such that $\dot{\sigma}_\zeta^2(h(u)+\rho) \leqslant \varepsilon(\sigma_\zeta^2(h(u)+\rho)/(h(u)+\rho))$.

Combination of (2.3) with the fact that $\sigma_\zeta(t)/t$ is regularly varying at ∞ with index $(\alpha_\infty/2) - 1 < 0$ implies that in order to complete the proof it is enough to show that

$$\limsup_{x\to 0} \frac{|x|}{\sigma_\zeta(x)} < \infty. \tag{2.4}$$

Since

$$\sigma_\zeta(1)\sigma_\zeta(x) \geqslant \mathrm{Cov}(\zeta(1),\zeta(x)) = (\sigma_\zeta^2(1)+\sigma_\zeta^2(x)-\sigma_\zeta^2(|1-x|))/2,$$

then for sufficiently small $x > 0$

$$\sigma_\zeta^2(1)-\sigma_\zeta^2(1-x) \leqslant 2\sigma_\zeta(1)\sigma_\zeta(x). \tag{2.5}$$

From the mean value theorem for each x there exists $\rho_x \in [0,x]$ such that $\sigma_\zeta^2(1) - \sigma_\zeta^2(1-x) = x\dot{\sigma}_\zeta^2(1-\rho_x)$. Combining it with (2.5), for sufficiently small $x > 0$, we get $x/\sigma_\zeta(x) \leqslant 4\sigma_\zeta(1)/\dot{\sigma}_\zeta^2(1)$, which implies (2.4).

This completes the proof. □

Remark 2.2. Families of Gaussian processes considered in Example 2.2 appeared in the analysis of some Gaussian fluid models (Massoulie and Simonian, 1997). Logarithmic asymptotics of supremum of such families of Gaussian processes was obtained by Dębicki (1999).

We need the following notation. Let $\bar{X}_{\eta_1;u}(t)$, $\bar{X}_{\eta_2;u}(t),\ldots,\bar{X}_{\eta_N;u}(t)$ be independent families of centered Gaussian processes that satisfy D0 with common $\Delta(u) = T > 0$ and $f(u)$. Define

$$\bar{X}_{\eta_1,\ldots,\eta_N;u}(t_1,\ldots,t_N) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{X}_{\eta_i;u}(t_i).$$

Theorem 2.1. *Let $n(u)$ be such that $\lim_{u\to\infty} n(u) = \infty$ and $\lim_{u\to\infty} f(u)/n(u) = 1$. Then*

$$\mathbb{P}\left(\sup_{(t_1,\ldots,t_N)\in[0,T]^N}\bar{X}_{\eta_1,\ldots,\eta_N;u}(t_1,\ldots,t_N) > n(u)\right) = \mathscr{H}_{\eta_1,\ldots,\eta_N}(T)\Psi(n(u))(1+\mathrm{o}(1)) \quad \text{as } u\to\infty. \tag{2.6}$$

Proof. We present the proof of Theorem 2.1 in Section 6.1. □

3. Generalized Pickands constants

In this section we define and study properties of *generalized Pickands constants.* We begin with a subadditivity property of $\mathscr{H}_{\eta_1,\ldots,\eta_N}(T)$.

Lemma 3.1. *If $\eta_1(t),\ldots,\eta_N(t)$ are independent centered Gaussian processes with stationary increments that satisfy* C1–C2, *then for all $T\in\mathbb{N}$*

$$\mathscr{H}_{\eta_1,\ldots,\eta_N}(T) \leqslant T^N\mathscr{H}_{\eta_1,\ldots,\eta_N}(1). \tag{3.1}$$

Proof. The complete proof is presented in Section 6. □

In the rest of this section we concentrate on the one-dimensional case of $\mathscr{H}_\eta(T)$. Note that the same argument as in the proof of Lemma 3.1 yields $\mathscr{H}_\eta(x+y) \leqslant \mathscr{H}_\eta(x)+\mathscr{H}_\eta(y)$ for all $x, y > 0$.

The main result of this section is given in the following theorem.

Theorem 3.1. *If the variance function $\sigma_\eta^2(t)$ of a centered Gaussian process $\eta(t)$ with stationary increments satisfies* C1–C2, *then*

$$\lim_{T\to\infty} \frac{\mathscr{H}_\eta(T)}{T} = \mathscr{H}_\eta, \tag{3.2}$$

where $\mathscr{H}_\eta > 0$ and is finite.

Proof. The proof of Theorem 3.1 is given in Section 6.2. □

If $\eta(t) = B_{\alpha/2}(t)$ is a fractional Brownian motion with Hurst parameter $\alpha/2$ ($\alpha \in (0,2)$), then it is known that Theorem 3.1 holds (see Piterbarg, 1996, p. 16, Theorem D.2). $\mathscr{H}_{B_{\alpha/2}}$ are known in the literature as the *Pickands constants*.

By the *generalized Pickands constants* we mean the constants $\mathscr{H}_\eta$ introduced in Theorem 3.1.

In the following theorem we give a criterion that enables us to compare the generalized Pickands constants $\mathscr{H}_\eta$.

Theorem 3.2. *Let $\eta_1(t), \eta_2(t)$ be centered Gaussian processes with stationary increments that satisfy* C1–C2. *If for all $t \geqslant 0$*

$$\sigma_{\eta_1}^2(t) \leqslant \sigma_{\eta_2}^2(t), \tag{3.3}$$

then

$$\mathscr{H}_{\eta_1} \leqslant \mathscr{H}_{\eta_2}. \tag{3.4}$$

Proof. The complete proof is presented in Section 6.3. □

Remark 3.1. Observe that the conclusion of Theorem 3.2 holds also for $\eta_2 = B_1(t)$ (that is for $\sigma_\eta^2(t) = t^2$). The proof of this fact is analogous to the proof of Theorem 3.2 with the exception that instead of $\bar{X}_{\eta_2;u}^{(\delta)}(t)$ we take $X((1+\delta)t/(\sqrt{2}u))$, where $X(t)$ is a stationary Gaussian process with covariance function $R(t) = \exp(-|t|^2)$.

Corollary 3.1. *If the variance function $\sigma_\eta^2(t)$ of $\eta(t) = \int_0^t Z(s)\,\mathrm{d}s$ satisfies* C1–C2, *where $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t)$, then*

$$\mathscr{H}_\eta \leqslant \sqrt{\frac{R(0)}{\pi}}.$$

Proof. Note that

$$\sigma_\eta^2(t) = \int_0^t \int_0^t \mathrm{Cov}(Z(v), Z(w))\,\mathrm{d}v\,\mathrm{d}w \leqslant R(0)t^2 = R(0)\sigma_{B_1}^2(t).$$

Thus from Theorem 3.2 and Remark 3.1 $\mathscr{H}_\eta \leqslant \mathscr{H}_{\sqrt{R(0)}B_1}$. Since $\mathscr{H}_{\sqrt{R(0)}B_1}(T) = \mathscr{H}_{B_1}(\sqrt{R(0)}T)$, then $\mathscr{H}_{\sqrt{R(0)}B_1} = \sqrt{R(0)}\mathscr{H}_{B_1}$. Hence

$$\mathscr{H}_\eta \leqslant \mathscr{H}_{\sqrt{R(0)}B_1} = \sqrt{R(0)}\mathscr{H}_{B_1} = \sqrt{\frac{R(0)}{\pi}}, \tag{3.5}$$

where (3.5) follows from the fact that $\mathscr{H}_{B_1} = 1/\sqrt{\pi}$. This completes the proof. □

In the following corollary we find an upper bound for $\mathscr{H}_\eta$ in the case of $\eta(t)$ with covariance function $\sigma_\eta^2(t)$ fulfilling some integrability conditions.

Corollary 3.2. *If* $\eta(t) = \int_0^t Z(s)\,\mathrm{d}s$ *satisfies* SRD.1, SRD.3 *and* $R(t) \geqslant 0$ *for each* $t \geqslant 0$, *where* $Z(t)$ *is a centered stationary Gaussian process with covariance function* $R(t)$, *then*

$$\mathscr{H}_\eta \leqslant 2\int_0^\infty R(s)\,\mathrm{d}s.$$

Proof. Let $\Upsilon = 2\int_0^\infty R(v)\,\mathrm{d}v$. From SRD.1, SRD.3 and the fact that $R(t) \geqslant 0$ for each $t \geqslant 0$ we infer that

$$\sigma_\eta^2(t) = 2\int_0^t \int_0^s R(v)\,\mathrm{d}v\,\mathrm{d}s$$

$$= \Upsilon t - 2\int_0^\infty vR(v)\,\mathrm{d}v + 2\int_t^\infty (v-t)R(v)\,\mathrm{d}v \tag{3.6}$$

$$\leqslant \Upsilon t = \sigma_{\sqrt{\Upsilon}B_{1/2}}^2(t) \tag{3.7}$$

and $\eta(t)$ satisfies C1–C2 with $\alpha_0 = 2$ and $\alpha_\infty = 1$.

Analogous considerations as in the proof of Corollary 3.1 yield

$$\mathscr{H}_{\sqrt{\Upsilon}B_{1/2}}(T) = \mathscr{H}_{B_{1/2}}(\Upsilon T). \tag{3.8}$$

Since $\mathscr{H}_{B_{1/2}} = 1$, then combining (3.8) with (3.7) and Theorem 3.2 we complete the proof. □

4. Double sum method

Theorem 2.1 enables us to obtain exact asymptotics for some families of Gaussian processes with variance function that attains its maximum at a unique point.

For the introduced in Section 2 family $\{X_{\eta;u}(t); u > 0\}$ of centered Gaussian processes we additionally assume that for sufficiently large $u > 0$ the function $\sigma_{X_{\eta;u}}(t)$

attains its maximum at a unique point t_u with $0 < t_u < \infty$. Without loss of generality we assume $\sigma^2_{X_{\eta;u}}(t_u) = 1$. Furthermore we claim that $\{X_{\eta;u}(t); u > 0\}$ satisfies the following conditions.

D1 Condition D0 is fulfilled for $(t,s) := (t + t_u, s + t_u)$.

D2 There exist $\beta > 0$ and a function $g(u)$ such that

$$\sup_{s,t \in J(u)} \left| \frac{1 - \sigma_{X_{\eta;u}}(t + t_u)}{|t|^\beta / g^2(u)} - 1 \right| \to 0 \quad \text{as } u \to \infty.$$

D3 $f(u)/g(u) \to 0$ as $u \to \infty$.

Theorem 4.1. *If the family* $\{X_{\eta;u}(t)\}$ *satisfies* D1–D3 *with* $\Delta(u) = (g(u)\log(n(u))/n(u))^{2/\beta}$, *where* $n(u)$ *is such that* $\lim_{u\to\infty} n(u) = \infty$ *and* $\lim_{u\to\infty} f(u)/n(u) = 1$, *then*

$$\mathbb{P}\left(\sup_{t \in J(u)} X_{\eta;u}(t + t_u) > n(u)\right) = \frac{2\mathscr{H}_\eta \Gamma(1/\beta)}{\beta} (A(u))^{-1} \Psi(n(u))(1 + o(1)) \tag{4.1}$$

as $u \to \infty$ *and* $A(u) = (n(u)/g(u))^{2/\beta}$.

Proof. The proof is given in Section 6.4. □

Remark 4.1. Note that, under conditions of Theorem 4.1, if $J(u) = [0, (g(u)\log(n(u))/n(u))^{2/\beta}]$, then $\mathbb{P}(\sup_{t\in J(u)} X_{\eta;u}(t + t_u) > n(u)) = \mathscr{H}_\eta(\Gamma(1/\beta)/\beta)(A(u))^{-1} \Psi(n(u))(1 + o(1))$ as $u \to \infty$.

In the next theorem we present a special case of Theorem 4.1, where we assume that in condition D1 we have $\eta(t) = B_{\alpha/2}(t)$ for $\alpha \in (0, 2]$. The property of multiplicativity of the variance function $\sigma^2_{B_{\alpha/2}}(t) = t^\alpha$ of fractional Brownian motion $B_{\alpha/2}(t)$ enables us to relax the assumption that $f(u)$ in D1 is of the same order as $n(u)$.

Theorem 4.2. *If the family* $\{X_{\eta;u}(t)\}$ *satisfies* D1–D2 *with* $\eta(t) = B_{\alpha/2}(t)$ *for* $\alpha \in (0, 2]$ *and* $\Delta(u) = (g(u)\log(n(u))/n(u))^{2/\beta}$, *where* $n(u)$ *is such that* $\lim_{u\to\infty} n(u) = \infty$ *and* $\lim_{u\to\infty} (n(u)/g(u))^{1/\beta}(f(u)/n(u))^{1/\alpha} = 0$, *then*

$$\mathbb{P}\left(\sup_{t \in J(u)} X_{\eta;u}(t + t_u) > n(u)\right) = \frac{2\mathscr{H}_{B_{\alpha/2}} \Gamma(1/\beta)}{\beta} \left(\frac{g(u)}{n(u)}\right)^{2/\beta} \left(\frac{n(u)}{f(u)}\right)^{2/\alpha} \Psi(n(u))(1 + o(1))$$

as $u \to \infty$.

Proof. The proof is presented in Section 6.4 after the proof of Theorem 4.1. □

Remark 4.2. If D1–D2 are satisfied with $f(u), g(u)$ being constant functions, then combination of $\lim_{u\to\infty} (n(u)/g(u))^{1/\beta}(f(u)/n(u))^{1/\alpha} = 0$ with $\lim_{u\to\infty} n(u) = \infty$ implies $\alpha < \beta$. In this case Theorem 4.2 is a part of Theorem 1 in Piterbarg and Prisyazhnyuk (1978).

5. Exact asymptotics of $\mathbb{P}(\sup_{t\geqslant 0}\int_0^t Z(s)\mathrm{d}s - ct > u)$

In this section we find the exact asymptotics of $\psi(u)=\mathbb{P}(\sup_{t\geqslant 0}\zeta(t)-ct>u)$ for the SRD model. Let $G=1/\int_0^\infty R(t)\,\mathrm{d}t$ and $B=\int_0^\infty tR(t)\,\mathrm{d}t$.

Theorem 5.1. *If $\zeta(t)$ possesses the* SRD *property, then*

$$\psi(u)=\frac{\mathscr{H}_{(cG/\sqrt{2})\zeta}}{Gc^2}\mathrm{e}^{-c^2G^2B}\mathrm{e}^{-Gcu}(1+\mathrm{o}(1)) \quad \text{as } u\to\infty. \tag{5.1}$$

Proof. The proof of Theorem 5.1 is presented in Section 6.5. □

Remark 5.1. Since $\sigma_\zeta^2(t)=2\int_0^t R(s)\,\mathrm{d}s$, then SRD.2 is equivalent to the fact that $\sigma_\zeta^2(t)$ is strictly increasing. It ensures that $\mathscr{H}_{(cG/\sqrt{2})\zeta}$ exists (Theorem 3.1) and assumption D1 of Theorem 4.1 is satisfied. In the language of the spectral density function $f_R(t)$ of the covariance function $R(t)$ we have

$$\begin{aligned}\int_0^t R(s)\,\mathrm{d}s &= 2\int_0^t\int_0^\infty \cos(xs)f_R(x)\,\mathrm{d}x\,\mathrm{d}s\\ &=2\int_0^\infty \frac{\sin(xt)}{x}f_R(x)\,\mathrm{d}x\end{aligned} \tag{5.2}$$

Hence if $0<f_R(0)<\infty$ and $f_R(x)/x$ is non increasing for $x\geqslant 0$, then from (5.2) assumption SRD.2 is satisfied. Moreover $G=1/(\pi f_R(0))$.

Remark 5.2. Using Corollary 3.1 we are able to give an asymptotical upper bound:

$$\limsup_{u\to\infty}\frac{\mathbb{P}(\sup_{t\geqslant 0}\zeta(t)-t>u)}{\sqrt{R(0)/2\pi}\,\mathrm{e}^{-G^2B}\mathrm{e}^{-Gu}}\leqslant 1. \tag{5.3}$$

This result is consistent with the asymptotical upper bound obtained by Dębicki and Rolski (1995).

6. Proofs

In this section we prove theorems presented in Sections 2–5.

6.1. Proof of Theorem 2.1

The idea of the proof is analogous to the proof of Pickands lemma presented in Piterbarg (1996, Lemma D.1) and is based on the fact that

$$\mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in[0,T]^N}\bar{X}_{\eta_1,\dots,\eta_N;u}(t_1,\dots,t_N)>n(u)\right)$$
$$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-v^2/2)$$

$$\times \mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in[0,T]^N} \bar{X}_{\eta_1,\dots,\eta_N;u}(t_1,\dots,t_N) > n(u) | \bar{X}_{\eta_1,\dots,\eta_N;u}(0,\dots,0) = v \right) \mathrm{d}v$$

$$= \Psi(n(u))(1+\mathrm{o}(1)) \int_{\mathbb{R}} \exp\left(\omega - \frac{\omega^2}{2n^2(u)}\right)$$

$$\times \mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in[0,T]^N} \xi_u(t_1,\dots,t_N) > \omega \,\middle|\, \bar{X}_{\eta_1,\dots,\eta_N;u}(0,\dots,0) = \frac{n^2(u)-\omega}{n(u)} \right) \mathrm{d}\omega, \tag{6.1}$$

and

$$\lim_{u\to\infty} \int_{\mathbb{R}} \exp\left(\omega - \frac{\omega^2}{2n^2(u)}\right)$$

$$\times \mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in[0,T]^N} \xi_u(t_1,\dots,t_N) > \omega | \bar{X}_{\eta_1,\dots,\eta_N;u}(0,\dots,0) = n(u) - \frac{\omega}{n(u)} \right) \mathrm{d}\omega$$

$$= \mathscr{H}_{\eta_1,\dots,\eta_N}(T), \tag{6.2}$$

where (6.1) is a consequence of changing of variables $v = n(u) - \omega/n(u)$ and the notation $\xi_u(t_1,\dots,t_N) = n(u)(\bar{X}_{\eta_1,\dots,\eta_N;u}(t_1,\dots,t_N) - n(u)) + \omega$. Equality (6.2) is a consequence of the fact that the family of processes

$$\chi_u(t_1,\dots,t_N) = \xi_u(t_1,\dots,t_N)| \left(\bar{X}_{\eta_1,\dots,\eta_N;u}(0,\dots,0) = n(u) - \frac{\omega}{n(u)}\right),$$
$$0 \leqslant t_1,\dots,t_N \leqslant T$$

converges weakly in $C[0,T]^N$ to the Gaussian process

$$\chi(t_1,\dots,t_N) = \sqrt{\frac{2}{N}} \sum_{i=1}^{N} \eta_i(t_i) - \frac{1}{N} \sum_{i=1}^{N} \sigma^2_{\eta_i}(t_i).$$

The proof of the weak convergence is analogous to the relevant part of the proof of Lemma D.1 in Piterbarg (1996) and is based on the convergence of finite dimensional distributions of the appropriate processes and tightness of family $\chi_u(t_1,\dots,t_N)$. In the sequel we argue that $\chi_u(t_1,\dots,t_N)$ is tight.

In order to prove the tightness of $\chi_u(t_1,\dots,t_N)$ it suffices to show that the sequence of centered processes $\chi_u^{(0)}(t_1,\dots,t_N) = \chi_u(t_1,\dots,t_N) - \mathbb{E}\chi_u(t_1,\dots,t_N)$ is tight. Since $\chi_u^{(0)}(0,\dots,0) = 0$ for all $u > 0$, then a straightforward consequence of Straf's criterion for tightness of Gaussian fields (Straf, 1972) implies that it suffices to show that for any $\mu, \varrho > 0$ there exists $\delta \in (0,1)$ and $u_0 > 0$ such that

$$\mathbb{P}\left(\sup_{\{(s_1,\dots,s_N): \|(s_1,\dots,s_N)-(t_1,\dots,t_N)\| \leqslant \delta\}} |\chi_u^{(0)}(s_1,\dots,s_N) - \chi_u^{(0)}(t_1,\dots,t_N)| \geqslant \mu \right) \leqslant \varrho\delta^N \tag{6.3}$$

for each $(t_1,\dots,t_N) \in [0,T]^N$ and $u > u_0$, where $\|(t_1,\dots,t_N)\| = \max_{i\in\{1,\dots,N\}} |t_i|$.

Note that for sufficiently large u

$$\mathbb{E}(\chi_u^{(0)}(t_1,\dots,t_N)-\chi_u^{(0)}(s_1,\dots,s_N))^2 \leqslant \sum_{i=1}^{N}\sigma_{\eta_i}^2(|t_i-s_i|) \leqslant C^2\sum_{i=1}^{N}|t_i-s_i|^{\alpha_{i,0}}$$

for all $(t_1,\dots,t_N),(s_1,\dots,s_N)\in[0,T]^N$, some constant $C>0$ and $\alpha_{i,0}$ being the indices of regularity at 0 of $\sigma_{\eta_i}^2(t)$ respectively. Thus

$$\max_{\{(s_1,\dots,s_N),(t_1,\dots,t_N):\|(s_1,\dots,s_N)-(t_1,\dots,t_N)\|\leqslant\delta\}} \operatorname{Var}(\chi_u^{(0)}(s_1,\dots,s_N)-\chi_u^{(0)}(t_1,\dots,t_N))$$
$$\leqslant C^2\sum_{i=1}^{N}|\delta|^{\alpha_{i,0}}.$$

which combined with Borell's theorem gives (6.3). □

6.2. Proof of Theorem 3.1

Before the proof of Theorem 3.1 we need some technical lemmas that are also of independent interest. We begin with the proof of Lemma 3.1.

Proof of Lemma 3.1. It suffices to note that under notation of Theorem 2.1, for sufficiently large u,

$$\mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in[0,T]^N}\bar{X}_{\eta_1,\dots,\eta_N;u}(t_1,\dots,t_N)>n(u)\right)$$
$$\leqslant\sum_{k_1=1}^{T}\dots\sum_{k_N=1}^{T}\mathbb{P}\left(\sup_{(t_1,\dots,t_N)\in\prod_{i=1}^{N}[k_i-1,k_i]}\bar{X}_{\eta_1,\dots,\eta_N;u}(t_1,\dots,t_N)>n(u)\right).$$

Now applying Theorem 2.1 to both sides of the above inequality we complete the proof. □

The following lemmas play a crucial role in sequel.

Lemma 6.1. *If the variance function $\sigma_\eta^2(t)$ of a centered Gaussian process $\eta(t)$ with stationary increments satisfies* C1–C2, *then for each $C>1$ there exists $\varepsilon>0$ such that*

$$\inf_{t>0}\frac{\sigma_\eta^2(Ct)}{\sigma_\eta^2(t)}\geqslant 1+\varepsilon.$$

Moreover for each $\varepsilon\in(0,1)$ there exists $C>1$ such that

$$\sup_{t>0}\frac{\sigma_\eta^2(t)}{\sigma_\eta^2(Ct)}\leqslant 1-\varepsilon.$$

Proof. The proof of Lemma 6.1 follows from assumption C2 that $\sigma_\eta^2(t)$ is regularly varying at 0 and at ∞ and the fact that $\sigma_\eta^2(t)$ is strictly increasing. □

Lemma 6.2. *If family* $\{\bar{X}_{\eta;u}(t); u>0\}$ *of centered Gaussian processes with continuous sample paths satisfies* D0 *with* $\Delta(u)$ *such that* $\lim_{u\to\infty}\Delta(u)=\infty$ *and*

$$\lim_{u\to\infty}\frac{\sigma_\eta^2(\Delta(u))}{f^2(u)}<1/2, \tag{6.4}$$

then for each $T>0$, $\delta>0$ *and* $n(u)$ *such that* $\lim_{u\to\infty} f(u)/n(u)=1$

$$\mathbb{P}\left(\sup_{s\in[0,T]}\bar{X}_{\eta;u}(s)>n(u);\ \sup_{t\in[\delta+T,\delta+2T]}\bar{X}_{\eta;u}(t)>n(u)\right)$$

$$\leqslant C_2T^2\exp(-C_1\sigma_\eta^2(\delta))\Psi(n(u))(1+\mathrm{o}(1)) \tag{6.5}$$

as $u\to\infty$. *Inequality* (6.5) *holds uniformly with respect to* u *for* $\delta\leqslant\Delta(u)-2T$.

Proof. The idea of the proof is analogous to the proof of Lemma 6.3 in Piterbarg (1996) and thus we present only the main steps of the argumentation.

Consider the Gaussian field $\boldsymbol{Y}_u(s,t)=\bar{X}_{\eta;u}(s)+\bar{X}_{\eta;u}(t)$ and let $A_0=[0,T]$, $A_{\delta+T}=[\delta+T,\delta+2T]$ for $0\leqslant\delta\leqslant\Delta(u)-2T$. We have

$$\mathbb{P}\left(\sup_{t\in[0,T]}\bar{X}_{\eta;u}(t)>n(u);\ \sup_{t\in[\delta+T,\delta+2T]}\bar{X}_{\eta;u}(t)>n(u)\right)$$

$$\leqslant\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}}\boldsymbol{Y}_u(s,t)>2n(u)\right). \tag{6.6}$$

Note that for each $s\in A_0$, $t\in A_{\delta+T}$ and sufficiently large u

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t))\geqslant 4-4\frac{\sigma_\eta^2(|t-s|)}{f^2(u)}\geqslant 2 \tag{6.7}$$

and

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t))\leqslant 4-\frac{\sigma_\eta^2(|t-s|)}{f^2(u)}\leqslant 4-\frac{\sigma_\eta^2(\delta)}{f^2(u)}, \tag{6.8}$$

where (6.7) follows from (6.4). Let $\bar{\boldsymbol{Y}}_u(s,t)=\boldsymbol{Y}_u(s,t)/\sqrt{\mathrm{Var}(\boldsymbol{Y}_u(s,t))}$ and observe that

$$\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}}\boldsymbol{Y}_u(s,t)>2n(u)\right)$$

$$\leqslant\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}}\bar{\boldsymbol{Y}}_u(s,t)>\frac{2n(u)}{\sqrt{4-\sigma_\eta^2(\delta)/f^2(u)}}\right). \tag{6.9}$$

Moreover for each s, $s_1\in A_0$ and t, $t_1\in A_{\delta+T}$

$$\mathbb{E}(\bar{\boldsymbol{Y}}_u(s,t)-\bar{\boldsymbol{Y}}_u(s_1,t_1))^2\leqslant\frac{4}{\mathrm{Var}(\boldsymbol{Y}_u(s,t))}\mathbb{E}(\boldsymbol{Y}_u(s,t)-\boldsymbol{Y}_u(s_1,t_1))^2$$

$$\leqslant 4(\mathbb{E}(\bar{X}_{\eta;u}(s)-\bar{X}_{\eta;u}(s_1))^2+\mathbb{E}(\bar{X}_{\eta;u}(t)-\bar{X}_{\eta;u}(t_1))^2)$$

$$\leqslant\frac{1}{2}(\mathbb{E}(\bar{X}_{\eta;u}(C_0s)-\bar{X}_{\eta;u}(C_0s_1))^2+\mathbb{E}(\bar{X}_{\eta;u}(C_0t)-\bar{X}_{\eta;u}(C_0t_1))^2), \tag{6.10}$$

where the existence of constant C_0 in (6.10) follows from Lemma 6.1. Hence for $X^{(1)}_{\eta;u}(t), X^{(2)}_{\eta;u}(t)$ being independent copies of the process $\bar{X}_{\eta;u}(t)$ the covariance function of the process $\frac{1}{\sqrt{2}}(X^{(1)}_{\eta;u}(C_0 s) + X^{(2)}_{\eta;u}(C_0 t))$ is dominated by the covariance function of $\bar{Y}_u(s,t)$. Thus from Slepian's inequality (see Piterbarg, 1996, Theorem C.1)

$$\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \bar{Y}_u(s,t) > \frac{2n(u)}{\sqrt{4-\sigma^2_\eta(\delta)/f^2(u)}}\right)$$

$$\leqslant \mathbb{P}\left(\sup_{(s,t)\in A_0^2} \frac{1}{\sqrt{2}}(X^{(1)}_{\eta;u}(C_0 s) + X^{(2)}_{\eta;u}(C_0 t)) > \frac{2n(u)}{\sqrt{4-\sigma^2_\eta(\delta)/f^2(u)}}\right) \tag{6.11}$$

$$= \mathscr{H}_{\eta,\eta}(C_0 T)\Psi\left(\frac{2n(u)}{\sqrt{4-\sigma^2_\eta(\delta)/f^2(u)}}\right)(1+\mathrm{o}(1)) \tag{6.12}$$

$$\leqslant C_2 T^2 \exp(-C_1\sigma^2_\eta(\delta))\Psi(n(u))(1+\mathrm{o}(1)), \tag{6.13}$$

where (6.11) holds uniformly with respect to u for $\delta \leqslant \Delta(u) - 2T$ and (6.12) follows from the combination of Theorem 2.1 and Lemma 3.1. Thus the assertion of Lemma 6.2 follows by combining (6.6), (6.9) and (6.13). □

Proof of Theorem 3.1. Since $\mathscr{H}_\eta(T)$ is subadditive, it suffices to prove that

$$\liminf_{T\to\infty} \frac{\mathscr{H}_\eta(T)}{T} > 0.$$

The above follows from the same argumentation, as in the proof of the existence of classical *Pickands constants* presented in Piterbarg (1996, the proof of Theorem D.2), applied to the family $X_{\eta;u}(t) = \eta(u+t)$. □

6.3. Proof of Theorem 3.2

Let $\delta > 0$ be given. Define

$$\bar{X}^{(\delta)}_{\eta_1;u}(t) = \frac{\eta_1(\sigma^{-1}_{\eta_1}(u) + (1+\delta)t)}{\sigma_{\eta_1}(\sigma^{-1}_{\eta_1}(u) + (1+\delta)t)}$$

$$\bar{X}^{(\delta)}_{\eta_2;u}(t) = \frac{\eta_2(\sigma^{-1}_{\eta_2}(u) + (1+\delta)t)}{\sigma_{\eta_2}(\sigma^{-1}_{\eta_2}(u) + (1+\delta)t)}$$

and observe that from C1–C2 the inverse functions $\sigma^{-1}_{\eta_1}(u), \sigma^{-1}_{\eta_2}(u)$ are well defined.

From Lemma 6.1 there exists $\varepsilon > 0$ such that

$$\sigma^2_{\eta_2}((1+\delta)t) \geqslant (1+\varepsilon)^2\sigma^2_{\eta_2}(t) \tag{6.14}$$

for each $t \geqslant 0$.

Let $T>0$ be given. From Lemma 2.1 processes $\bar{X}^{(\delta)}_{\eta_1;u}(t)$, $\bar{X}^{(\delta)}_{\eta_2;u}(t)$ satisfy D0 with $f(u)=\sqrt{2}u$, $\Delta(u)=T$ and $\eta=\eta_1$ or $\eta=\eta_2$, respectively. Thus for $s,t\in[0,T]$ and sufficiently large u

$$\begin{aligned}1-\mathrm{Cov}(\bar{X}^{(\delta)}_{\eta_2;u}(t),\bar{X}^{(\delta)}_{\eta_2;u}(s)) &\geqslant \frac{1}{1+\varepsilon}\frac{\sigma^2_{\eta_2}((1+\delta)|t-s|)}{2u^2}\\ &\geqslant (1+\varepsilon)\frac{\sigma^2_{\eta_2}(|t-s|)}{2u^2} \qquad (6.15)\\ &\geqslant (1+\varepsilon)\frac{\sigma^2_{\eta_1}(|t-s|)}{2u^2},\\ &\geqslant 1-\mathrm{Cov}(\bar{X}^{(0)}_{\eta_1;u}(t),\bar{X}^{(0)}_{\eta_1;u}(s)), \qquad (6.16)\end{aligned}$$

where (6.15) follows from (6.14) and (6.16) follows from the fact that $\sigma^2_{\eta_1}(t)\leqslant\sigma^2_{\eta_2}(t)$.

Hence for each $\delta>0$, $t>0$ and sufficiently large u we can apply Slepian's inequality

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,(1+\delta)T]}\bar{X}^{(0)}_{\eta_2;u}(t)>\sqrt{2}u\right) &= \mathbb{P}\left(\sup_{t\in[0,T]}\bar{X}^{(\delta)}_{\eta_2;u}(t)>\sqrt{2}u\right)\\ &\geqslant \mathbb{P}\left(\sup_{t\in[0,T]}\bar{X}^{(0)}_{\eta_1;u}(t)>\sqrt{2}u\right). \qquad (6.17)\end{aligned}$$

To complete the proof it is enough to note that from Theorem 2.1

$$\mathbb{P}\left(\sup_{t\in[0,(1+\delta)T]}\bar{X}^{(0)}_{\eta_2;u}(t)>\sqrt{2}u\right)=\mathscr{H}_{\eta_2}((1+\delta)T)\Psi(\sqrt{2}u)(1+o(1))$$

and

$$\mathbb{P}\left(\sup_{t\in[0,T]}\bar{X}^{(0)}_{\eta_1;u}(t)>\sqrt{2}u\right)=\mathscr{H}_{\eta_1}(T)\Psi(\sqrt{2}u)(1+o(1))$$

as $u\to\infty$. Combining this with (6.17) we obtain that $\mathscr{H}_{\eta_2}((1+\delta)T)\geqslant\mathscr{H}_{\eta_1}(T)$ for each $\delta>0$. Having in mind that $\mathscr{H}_{\eta_1}=\lim_{T\to\infty}\mathscr{H}_{\eta_1}(T)/T$ and $\mathscr{H}_{\eta_2}=\lim_{T\to\infty}\mathscr{H}_{\eta_2}(T)/T$ the proof is completed. □

6.4. Proof of Theorem 4.1

The idea of the proof of Theorem 4.1 is analogous to the proof of Theorem D.3 (Piterbarg, 1996) and thus we present only the main steps of the argumentation.

In the proof we denote for short $\theta(u)=\mathbb{P}(\sup_{t\in J(u)}X_{\eta;u}(t+t_u)>n(u))$. From D2 for each $\varepsilon\in(0,1)$ there exists u_0 such that for $u\geqslant u_0$ and $t\in J(u)$

$$\theta(u)\leqslant\mathbb{P}\left(\sup_{t\in J(u)}\bar{X}_{\eta;u}(t+t_u)\frac{1}{1+(1-\varepsilon)\frac{|t|^\beta}{g^2(u)}}>n(u)\right)=\theta_1(u)$$

and

$$\theta(u) \geqslant \mathbb{P}\left(\sup_{t\in J_u} \bar{X}_{\eta;u}(t+t_u)\frac{1}{1+(1+\varepsilon)\frac{|t|^\beta}{g^2(u)}} > n(u)\right) = \theta_2(u).$$

The rest of the proof consists of two parts, where an upper and lower bound for $\theta(u)$ is derived.

1°. (*Upper bound*) Our goal is to prove that

$$\limsup_{u\to\infty} \frac{\theta(u)}{2\mathscr{H}_\eta(\Gamma(1/\beta)/\beta)(A(u))^{-1}\Psi(n(u))} \leqslant 1. \tag{6.18}$$

Since $\theta(u) \leqslant \theta_1(u)$, we focus on the asymptotics of $\theta_1(u)$. Let $T>0$ be given and let $\Delta(u) = (g(u)\log(n(u))/n(u))^{2/\beta}$. Note that $\Delta(u) \to \infty$ as $u \to \infty$. We consider a skeleton $J_k = [kT, (k+1)T]$ of $\mathbb{R}$ and define events

$$C_k(u) = \begin{cases} \max_{t\in J_k}\{\bar{X}_{\eta;u}(t+t_u) > n(u)(1+(1-\varepsilon)|(k+1)T|^\beta/g^2(u))\} & k=-1,-2,\dots \\ \max_{t\in J_k}\{\bar{X}_{\eta;u}(t+t_u) > n(u)(1+(1-\varepsilon)|kT|^\beta/g^2(u))\} & k=0,1,\dots\,. \end{cases} \tag{6.19}$$

Now using the Bonferroni's inequality and Theorem 2.1 we get

$$\begin{aligned}
\theta_1(u) &\leqslant \sum_{-(\Delta(u)/T)-1\leqslant k\leqslant \Delta(u)/T} \mathbb{P}(C_k(u)) \\
&= \sum_{-(\Delta(u)/T)-1\leqslant k\leqslant 0} \mathscr{H}_\eta(T)\Psi\left(n(u)\left(1+(1-\varepsilon)\frac{|(k+1)T|^\beta}{g^2(u)}\right)\right)(1+o(1)) \\
&\quad + \sum_{0<k\leqslant \Delta(u)/T} \mathscr{H}_\eta(T)\Psi\left(n(u)\left(1+(1-\varepsilon)\frac{|kT|^\beta}{g^2(u)}\right)\right)(1+o(1)). \\
&\leqslant \frac{\mathscr{H}_\eta(T)\Psi(n(u))}{TA(u)} \sum_{-(\Delta(u)/T)-1\leqslant k\leqslant 0} TA(u)\exp(-(1-\varepsilon)(TA(u)|(k+1)|)^\beta) \\
&\qquad \times(1+o(1)) \\
&\quad + \frac{\mathscr{H}_\eta(T)\Psi(n(u))}{TA(u)} \sum_{0<k\leqslant \Delta(u)/T} TA(u)\exp(-(1-\varepsilon)(TA(u)k)^\beta)(1+o(1))
\end{aligned} \tag{6.20}$$

as $u\to\infty$, where $A(u)=(n(u)/g(u))^{2/\beta}$ Since $\lim_{u\to\infty} A(u)=0$ (see D3 and assumption that $\lim_{u\to\infty} f(u)/n(u) = 1$), then letting $u\to\infty$ and $T\to\infty$ in such a way that $TA(u)\to 0$, we obtain

$$\limsup_{u\to\infty} \frac{\theta_1(u)}{2\mathscr{H}_\eta(\Psi(n(u))/A(u))\int_0^\infty e^{-(1-\varepsilon)x^\beta}\,dx} \leqslant 1.$$

Using that $\beta\int_0^\infty e^{-x^\beta}\,dx = \Gamma(1/\beta)$ and letting $\varepsilon\to 0$ we obtain (6.18).

2°. (*Lower bound*) To get

$$\liminf_{u\to\infty} \frac{\theta(u)}{2\mathscr{H}_\eta(\Gamma(1/\beta)/\beta)(\Delta(u))^{-1}\Psi(n(u))} \geqslant 1 \tag{6.21}$$

we have to adapt the preceding proof as follows.

Define events

$$C'_k(u) = \begin{cases} \max_{t\in J_k} \{\bar{X}_{\eta;u}(t+t_u) > n(u)(1+(1+\varepsilon)|kT|^\beta/g^2(u))\} & k=-1,-2,\dots \\ \max_{t\in J_k} \{\bar{X}_{\eta;u}(t+t_u) > n(u)(1+(1+\varepsilon)|(k+1)T|^\beta/g^2(u))\} & k=0,1,\dots . \end{cases} \tag{6.22}$$

Using Bonferroni's inequality

$$\theta_2(u) \geqslant \sum_{-(\Delta(u)/T)\leqslant k\leqslant(\Delta(u)/T)-1} \mathbb{P}(C'_k(u))$$

$$-2 \sum_{-(\Delta(u)/T)\leqslant k<l\leqslant(\Delta(u)/T)-1} \mathbb{P}(C'_k(u)\cap C'_l(u)).$$

Thus it suffices to prove that

$$\lim_{u\to\infty} \frac{\sum_{-(\Delta(u)/T)\leqslant k<l\leqslant(\Delta(u)/T)-1} \mathbb{P}(C'_k(u)\cap C'_l(u))}{(\Delta(u))^{-1}\Psi(n(u))} = 0,$$

which, using Lemma 6.2, follows by the same argumentation as the estimation of the double sum in the proof of Theorem D.1 in Piterbarg (1996). This completes the proof. □

Proof of Theorem 4.2. The proof follows from the straightforward application of Theorem 4.1 to the family

$$Y_{\eta;u}(t+t_u) = X_{\eta;u}\left(\left(\frac{f(u)}{n(u)}\right)^{2/\alpha} t + t_u\right). \quad \square$$

6.5. Proof of Theorem 5.1

The idea of the proof of Theorem 5.1 is based on an appropriate application of Theorem 4.1. Namely since

$$\psi(u) = \mathbb{P}\left(\sup_{t\geqslant 0} \zeta(t) - ct > u\right) = \mathbb{P}\left(\sup_{t\geqslant 0} \frac{\zeta(t)}{c} - t > \frac{u}{c}\right)$$

it suffices to give the proof for $c=1$. Thus without loss of generality we assume that $c=1$.

We rewrite

$$\mathbb{P}\left(\sup_{t\geqslant 0}(\zeta(t)-t) > u\right) = \mathbb{P}\left(\sup_{t\geqslant 0} X_{\zeta;u}(t) > m(u)\right),$$

where $X_{\zeta;u}(t) = (\zeta(t)/(u+t))m(u)$ and $m(u) = \min_{t\geqslant 0}(u+t)/\sigma_\zeta(t)$.

Remark 6.1. Condition SRD yields

$$\sigma_\zeta^2(t) = 2\int_0^t\int_0^s R(v)\,dv\,ds = \frac{2}{G}t - 2B + r(t), \tag{6.23}$$

where

$$r(t) = 2\int_t^\infty (s-t)R(s)\,ds = o(t^{-1})$$

as $t \to \infty$ (see Ibragimov and Linnik, 1971). Hence

$$\dot{\sigma}_\zeta^2(t) = 2\int_0^t R(v)\,dv = \frac{2}{G} + o(1) \tag{6.24}$$

as $t \to \infty$ and $\ddot{\sigma}_\zeta^2(t) = 2R(t)$.

Lemma 6.3. *If the variance function* $\sigma_\zeta^2(t)$ *of the process* $\zeta(t)$ *satisfies* C1–C2, *then for*

$$\bar{X}_{\zeta;u}(t) = \frac{\zeta(h(u)+t)}{\sigma_\zeta(h(u)+t)}$$

where $h(u)$ *is such that* $\lim_{u\to\infty} h(u) = \infty$, *there exists constant* $C > 0$ *such that for each* $I_{\delta,T} = [\delta, \delta+T] \subset [-h(u)/2, h(u)]$ *and sufficiently large* u

$$\mathbb{P}\left(\sup_{t\in I_{\delta,T}} \bar{X}_{\zeta;u}(t) > w\right) \leqslant \mathbb{P}\left(\sup_{t\in I_{0,T}} \bar{X}_{\zeta;u}(Ct) > w\right) \tag{6.25}$$

for all $w > 0$.

Proof. The idea of the proof is based on Slepian's inequality. Let $s, t \in I_{0,T}$. Hence for sufficiently large u we have

$$s+\delta, t+\delta \in I_{\delta,T} \subset [-h(u)/2, h(u)]. \tag{6.26}$$

From the definition of $\bar{X}_{\zeta;u}(t)$, for sufficiently large u we have

$$\begin{aligned} &\mathbb{E}(\bar{X}_{\zeta;u}(t+\delta) - \bar{X}_{\zeta;u}(s+\delta))^2 \\ &\quad = 2(1 - \mathrm{Cov}(\bar{X}_{\zeta;u}(t+\delta), \bar{X}_{\zeta;u}(s+\delta))) \\ &\quad = \frac{\sigma_\zeta^2(|t-s|)}{\sigma_\zeta(h(u)+s+\delta)\sigma_\zeta(h(u)+t+\delta)} \\ &\qquad - \frac{(\sigma_\zeta(h(u)+t+\delta) - \sigma_\zeta(h(u)+s+\delta))^2}{\sigma_\zeta(h(u)+s+\delta)\sigma_\zeta(h(u)+t+\delta)}. \end{aligned} \tag{6.27}$$

From (6.26) it follows that $h(u)+s+\delta, h(u)+t+\delta > h(u)/2$ and since $\sigma_\zeta(t)$ is increasing, the expression in (6.27) is less or equal than $\sigma_\zeta^2(|t-s|)/(\sigma_\zeta(h(u)/2)\sigma_\zeta(h(u)/2))$. Now by

Remark 6.1 and Lemma 6.1, there exist constants $C_1, C_2 > 0$ such that

$$\frac{\sigma_\zeta^2(|t-s|)}{\sigma_\zeta(h(u)/2)\sigma_\zeta(h(u)/2)} \leqslant C_1 \frac{\sigma_\zeta^2(|t-s|)}{\sigma_\zeta(h(u))\sigma_\zeta(h(u))} \leqslant \frac{\sigma_\zeta^2(C_2|t-s|)}{\sigma_\zeta(h(u))\sigma_\zeta(h(u))}.$$

Furthermore, by Lemma 6.1 and Lemma 2.1, there exists constant $C > 0$ such that the above is less or equal to

$$2(1 - \mathrm{Cov}(\bar{X}_{\zeta;u}(Ct), \bar{X}_{\zeta;u}(Cs))) = \mathbb{E}(\bar{X}_{\zeta;u}(Ct) - \bar{X}_{\zeta;u}(Cs))^2.$$

Now it is enough to use Slepian's inequality to complete the proof. □

Lemma 6.4. *If $\zeta(t)$ possesses* SRD *property then, for sufficiently large u, function $s_u(t) = (u+t)^2/\sigma_\zeta^2(t)$ attains its minimum at the unique point t_u such that*

$$t_u = u(1 + o(1)) \quad \text{as} \quad u \to \infty. \tag{6.28}$$

Proof. First we prove the uniqueness of the maximum of $s_u(t)$. The idea of the proof is to show that for sufficiently large u the equation

$$s_u'(t) = 0 \tag{6.29}$$

is satisfied for exactly one point $t = t_u$.

Let $\phi(t) = 2(\sigma_\zeta^2(t)/\dot{\sigma}_\zeta^2(t)) - t$. Since $s_u'(t) = (2\sigma_\zeta^2(t)(u+t) - \dot{\sigma}_\zeta^2(t)(u+t)^2)/(u+t)^4$, then (6.29) is equivalent to the fact that, for each sufficiently large u, $\phi(t) = u$ holds for exactly one point $t = t_u$.

Thus, using that $\lim_{t\to 0}\phi(t) = 0$, it is enough to prove that $\phi(t)$ is ultimately strictly increasing and $\lim_{t\to\infty}\phi(t) = \infty$.

It follows from the fact that due to SRD (see also Remark 6.1) we have $\lim_{t\to\infty}\sigma_\zeta^2(t)\ddot{\sigma}_\zeta^2(t) = 0$ and hence

$$\lim_{t\to\infty}\phi'(t) = \lim_{t\to\infty}\left(1 - \frac{\sigma_\zeta^2(t)\ddot{\sigma}_\zeta^2(t)}{(\dot{\sigma}_\zeta^2(t))^2}\right) = 1. \tag{6.30}$$

In order to prove (6.28) note that from (6.23) and (6.24) we have $(\sigma_\zeta^2(t_u)/\dot{\sigma}_\zeta^2(t_u)) = t_u(1 + o(1))$ as $u \to \infty$. Since for the point t_u we have

$$\phi(t_u) = 2\frac{\sigma_\zeta^2(t_u)}{\dot{\sigma}_\zeta^2(t_u)} - t_u = u$$

then the proof of (6.28) is completed. □

In the sequel, t_u will denote the point at which $s_u(t) = (u+t)^2/\sigma_\zeta^2(t)$ attains its minimum.

Proposition 6.1. *If $\zeta(t)$ possesses* SRD *property, then*

$$m^2(u) = 2Gu + 2G^2B + o(1) \quad \text{as } u \to \infty. \tag{6.31}$$

Proof. The idea of the proof is to find asymptotical upper and lower bounds for $m(u)$ as $u \to \infty$.

Let $\delta \in (0,1)$ be given. From Lemma 6.4 for sufficiently large u

$$m(u) = \min_{t\in[(1-\delta)t_u,(1+\delta)t_u]} \frac{(u+t)^2}{\sigma_\zeta^2(t)}. \tag{6.32}$$

Following Remark 6.1 and Lemma 6.4, for sufficiently large u and $t \in [(1-\delta)t_u,(1+\delta)t_u]$, we have

$$\sigma_\zeta^2(t) \leqslant \frac{2}{G}t - 2B + \frac{2}{u} \tag{6.33}$$

$$\sigma_\zeta^2(t) \geqslant \frac{2}{G}t - 2B - \frac{2}{u}. \tag{6.34}$$

Combining (6.32) with (6.33) and (6.34) we obtain

$$m(u) \leqslant \min_{t\in[(1-\delta)t_u,(1+\delta)t_u]} \frac{(u+t)^2}{(2/G)t - 2B - 2/u} = 2Gu + 2BG^2 + \frac{2G^2}{u}$$

$$m(u) \geqslant \min_{t\in[(1-\delta)t_u,(1+\delta)t_u]} \frac{(u+t)^2}{(2/G)t - 2B + 2/u} = 2Gu + 2BG^2 - \frac{2G^2}{u}$$

for sufficiently large u.

This completes the proof. □

Lemma 6.5. *If $\zeta(t)$ possesses* SRD *property, then the family $X_{\zeta;u}(t)=(\zeta(t)/(u+t))m(u)$ fulfills conditions* D1–D2 *with $\beta = 2$, $g(u) = \sqrt{2}(u+t_u)$, $f(u) = G\sigma_\zeta(t_u)$, $\eta(t) = (G/\sqrt{2})\zeta(t)$ and*

$$J(u) = [-\Delta(u), \Delta(u)], \tag{6.35}$$

where $\Delta(u) = (g(u)\log(m(u))/m(u))^{2/\beta}$.

Proof. Note that $\bar{X}_{\zeta;u}(t+t_u) = \zeta(t+t_u)/\sigma_\zeta(t+t_u)$ and $(\Delta(u)/t_u) \to 0$. Thus D1 is satisfied for $f(u) = G\sigma_\zeta(t_u)$ and $\eta(t) = (G/\sqrt{2})\zeta(t)$. Moreover

$$\sigma_{X_{\zeta;u}}(t+t_u) = m(u)\frac{\sigma_\zeta(t+t_u)}{u+t+t_u}.$$

Hence

$$\vartheta(u,t) = \sigma_{X_{\zeta;u}}(t+t_u) - 1$$

$$= \frac{\frac{1}{2}t^2\ddot{\sigma}_\zeta(t_u+\theta)(u+t_u)}{\sigma_\zeta(t_u)(u+t_u+t)}, \tag{6.36}$$

where (6.36) follows from the expansion of $\sigma_\zeta(t+t_u)$ into a Taylor series with respect to t where $\theta \in [-\Delta(u), \Delta(u)]$. Since $\sigma_\zeta(t) \in C^2$ (see Remark 6.1) and

$$\ddot{\sigma}_\zeta(x) = \frac{\ddot{\sigma}_\zeta^2(x)}{2\sigma_\zeta(x)} - \frac{1}{4}\frac{(\dot{\sigma}_\zeta^2(x))^2}{\sigma_\zeta^3(x)},$$

then dividing (6.36) by t^2 we obtain

$$\frac{\vartheta(u,t)}{t^2} = \frac{\ddot{\sigma}_\zeta^2(t_u+\theta)(u+t_u)}{4\sigma_\zeta(t_u+\theta)\sigma_\zeta(t_u)(u+t_u+t)} - \frac{1}{8}\frac{(\dot{\sigma}_\zeta^2(t_u+\theta))^2(u+t_u)}{\sigma_\zeta^3(t_u+\theta)\sigma_\zeta(t_u)(u+t_u+t)}$$
$$= S_1 - S_2.$$

Now from Remark 6.1 we immediately get that $S_2/(\sqrt{2}(u+t_u)) \to 1$ as $u \to \infty$, uniformly with respect $t, \theta \in [-\Delta(u), \Delta(u)]$. Moreover uniformly for $\theta \in [-\Delta(u), \Delta(u)]$

$$\frac{S_1}{S_2} = \frac{2\ddot{\sigma}_\zeta^2(t_u+\theta)\sigma_\zeta^3(t_u+\theta)}{\sigma_\zeta(t_u+\theta)(\dot{\sigma}_\zeta^2(t_u+\theta))^2} = \frac{2\ddot{\sigma}_\zeta^2(t_u+\theta)\sigma_\zeta^2(t_u+\theta)}{(\dot{\sigma}_\zeta^2(t_u+\theta))^2} \to 0 \tag{6.37}$$

as $u \to \infty$, where (6.37) is a consequence of SRD. This completes the proof. □

Lemma 6.6. *If $\zeta(t)$ possesses* SRD *property, then for $J(u)$ defined by* (6.35)

$$\mathbb{P}\left(\sup_{t\in[0,\infty)} X_{\zeta;u}(t) > m(u)\right) = \mathbb{P}\left(\sup_{t\in J(u)} X_{\zeta;u}(t+t_u) > m(u)\right)(1+\mathrm{o}(1)) \tag{6.38}$$

as $u \to \infty$.

Proof. To prove (6.38) it is sufficient to show that

$$\mathbb{P}\left(\sup_{t\in[-t_u,\infty)\setminus J(u)} X_{\zeta;u}(t+t_u) > m(u)\right) = \mathrm{o}(\Psi(m(u))) \tag{6.39}$$

as $u \to \infty$. Let $\Delta(u)$ and $J(u)$ be the same as defined in Lemma 6.5. We have

$$\mathbb{P}\left(\sup_{t\in[-t_u,\infty)\setminus J(u)} X_{\zeta;u}(t+t_u) > m(u)\right)$$
$$\leqslant \mathbb{P}\left(\sup_{t\in[-t_u,-t_u/2]} X_{\zeta;u}(t+t_u) > m(u)\right)$$
$$+\mathbb{P}\left(\sup_{t\in[-t_u/2,-\Delta(u)]\cup[\Delta(u),t_u]} X_{\zeta;u}(t+t_u) > m(u)\right)$$
$$+\mathbb{P}\left(\sup_{t\in[t_u,\infty)} X_{\zeta;u}(t+t_u) > m(u)\right).$$

Let $\sigma_{X_{\zeta;u}}(A) = \max_{t\in[-t_u,\infty)\setminus J(u)} \sigma_{X_{\zeta;u}}(t+t_u)$. Note that from Lemma 6.4 and Lemma 6.5 for sufficiently large u

$$\sigma_{X_{\zeta;u}}(A) \leqslant 1 - \frac{\Delta^2(u)}{2(u+t_u)^2} \leqslant \frac{1}{1+\log^2(m(u))/m^2(u)},$$

From Lemma 6.3 there exists $C>0$ such that for sufficiently large u and $[i,i+1] \subset \{[-t_u/2, -\Delta(u)] \cup [\Delta(u), t_u]\}$ we have

$$\mathbb{P}\left(\sup_{t\in[i,i+1]} \bar{X}_{\zeta;u}(t+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)$$
$$\leqslant \mathbb{P}\left(\sup_{t\in[0,1]} \bar{X}_{\zeta;u}(Ct+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right).$$

Hence

$$\mathbb{P}\left(\sup_{t\in[-t_u/2,-\Delta(u)]\cup[\Delta(u),t_u]} X_{\zeta;u}(t+t_u) > m(u)\right)$$
$$\leqslant \sum_{\Delta(u)-1\leqslant i\leqslant \frac{t_u}{2}+1} \mathbb{P}\left(\sup_{t\in[-i,-i+1]} \bar{X}_{\zeta;u}(t+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)$$
$$+ \sum_{\Delta(u)-1\leqslant i\leqslant t_u+1} \mathbb{P}\left(\sup_{t\in[i,i+1]} \bar{X}_{\zeta;u}(t+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)$$
$$\leqslant t_u \mathbb{P}\left(\sup_{t\in[0,1]} \bar{X}_{\zeta;u}(Ct+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)$$
$$= t_u \,\mathrm{Const}\, \Psi\left(m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)(1+o(1)) = o(\Psi(m(u))). \quad (6.40)$$

The proof of

$$\mathbb{P}\left(\sup_{t\in[-t_u,-t_u/2]} X_{\zeta;u}(t+t_u) > m(u)\right) + \mathbb{P}\left(\sup_{t\in[t_u,\infty)} X_{\zeta;u}(t+t_u) > m(u)\right)$$
$$= o(\Psi(m(u)) \quad (6.41)$$

follows in a straightforward way from Borell's inequality and the fact that

$$\sup_{t\in[-t_u,-t_u/2]\cup[t_u,\infty)} \sigma^2_{X_{\zeta;u}}(t+t_u) \leqslant 1 - \mathrm{Const}_2,$$

where $\mathrm{Const}_2 > 0$ is a constant. Thus (6.40) combined with (6.41) completes the proof. □

Proof of Theorem 5.1. From Lemma 6.6 we have

$$\mathbb{P}\left(\sup_{t\geqslant 0}(\zeta(t)-t)>u\right)=\mathbb{P}\left(\sup_{t\geqslant 0}X_{\zeta;u}(t)>m(u)\right)$$
$$=\mathbb{P}\left(\sup_{t\in J(u)}X_{\zeta;u}(t)>m(u)\right)(1+\mathrm{o}(1)).$$

Thus

$$\mathbb{P}\left(\sup_{t\geqslant 0}(\zeta(t)-t)>u\right)$$
$$=\frac{2\mathscr{H}_{(G/\sqrt{2})\zeta}\Gamma(1/2)}{2}\left(\frac{m(u)}{\sqrt{2}(u+t_u)}\right)^{-2/\beta}\Psi(m(u))(1+\mathrm{o}(1)) \tag{6.42}$$
$$=\sqrt{\pi}\mathscr{H}_{(G/\sqrt{2})\zeta}\frac{2\sqrt{u}}{\sqrt{G}}\Psi(m(u))(1+\mathrm{o}(1))$$
$$=\frac{\mathscr{H}_{(G/\sqrt{2})\zeta}}{G}\mathrm{e}^{-G^2B}\mathrm{e}^{-Gu}(1+\mathrm{o}(1)), \tag{6.43}$$

where (6.42) and (6.43) follow from Lemma 6.5 and Theorem 4.2 and the fact that $\Gamma(1/2)=\sqrt{\pi}$. This completes the proof. □

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